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Measuring 2-D Shape Compactness Using the Contact Perimeter

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Abstract—A new perimeter for shapes composed of cells is defined. This perimeter is called the *contact perimeter*, which corresponds to the sum of the boundaries of neighboring cells of the shape. Also, a relation between the perimeter of the shape and the contact perimeter is presented. The contact perimeter corresponds to the measure of compactness proposed here called *discrete compactness*. In this case, the term compactness does not refer to point-set topology, but is related to intrinsic properties of objects.

Keywords—Contact perimeter, Measure of compactness, Discrete compactness, Shape properties, Shape classification, Shape analysis.

1. INTRODUCTION

The study of shape properties is an important part in computer vision. The main properties for planar shapes are *area* and *perimeter*, which are basic descriptive properties; the *eccentricity* or *elongation*, which is the ratio of the maximum axis to minimum axis; the *principal axes* [1]; the *moments* [2]; The *Euler number*, which is a topological property and is defined as (number of connected regions)—(number of holes) [3]; *compactness* [4]; the *slope density function* [5]; the *concavity tree* [6]; the *shape numbers* [7], which are a measure of shape similarity and are related to the resolution of the digitalization scheme.

Compactness plays an important role in classification and shape analysis. Nevertheless, when we use the measure of classical compactness in the digital domain, we find some problems, which will be analyzed in the content of this paper. In this work, we present an approach for measuring the compactness of objects composed of a finite number of cells; we define the contact perimeter and its relation to the perimeter of the shape. This paper is organized as follows. In Section 2, we present the measure of classical compactness. In Section 3, we define the contact perimeter and the relation between perimeters for shapes. In Section 4, we give the measure of discrete compactness. Section 5 gives some results using shapes of the real world and finally, in Section 6, we give some conclusions.

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2. THE MEASURE OF CLASSICAL COMPACTNESS

The *compactness* C of an object can be measured by the ratio (perimeter^2)/area, which is dimensionless and minimized by a disk [3]. The measure of compactness is an intrinsic property of objects [4]. Therefore, the measure of compactness is invariant under geometric transformations such as: translation, rotation, and scaling. In the digital domain, most shapes have no well-defined contours, that is due to the noise of the input devices used, such as: vidicons, CCD cameras, scanners, sensors, or analog-to-digital converters.

The above-mentioned devices may produce noisy contours and consequently larger perimeters, which will affect the measure of compactness. An advantage of using the measure here proposed of discrete compactness is that the accuracy of the shape contour and the shape area is measured using the same element (the cell). Figure 1 shows some problems which arise when determining compactness of shapes using the classical measure in the digital domain. Figure 1a presents a circle having a well-defined contour, in this case its measure of compactness is 12.56. However, in Figure 1b, we show the same circle holding a noisy contour with increased measure of 29.67, which corresponds to the measure of the shape in Figure 1c. Another case, Figure 1d presents the snowflake curve, which produces an aberrant measure when its perimeter grows to larger values and its area holds a discrete value.

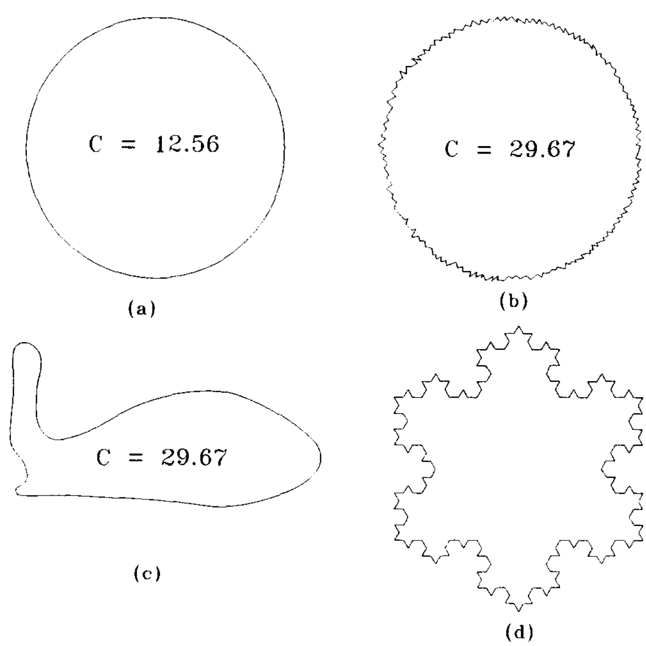


Figure 1. Measures of classical compactness: (a) a circle having a well-defined contour; (b) a circle having a noisy contour; (c) a shape holding the same measure as in (b); (d) the snowflake shape.

3. CONCEPTS AND DEFINITIONS

An important simplification in this work is the assumption that an *entity* has been isolated from the real world. This is called the *shape*, and is defined as a result of previous processing. Figure 2a shows a shape composed of cells, in this case pixels. In the content of this work, the length of all the sides of cells is considered equal to one. In order to introduce the proposed compactness measurement method, a number of geometrical concepts are defined below.

3.1. Perimeters

In this paper, we define the contact perimeter for each shape composed of cells. Also, we define the relation between the contact perimeter and the perimeter of the shape. This definition of

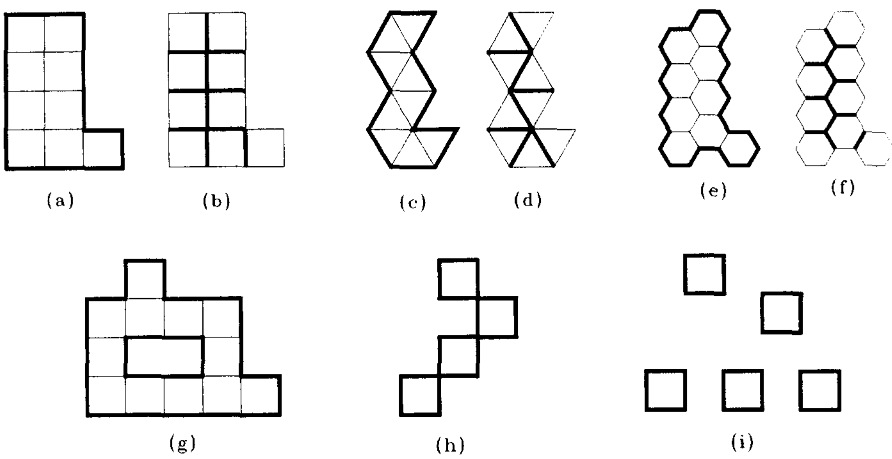


Figure 2. The perimeters and their relations: (a) the perimeter of a shape using the rectangular tesselation; (b) the contact perimeter; (c) the perimeter of a shape using the triangular tesselation; (d) the contact perimeter; (e) the perimeter of a shape using the hexagonal tesselation; (f) the contact perimeter; (g) a shape with a hole; (h) a shape without contact perimeter; (i) a shape without contact perimeter.

perimeter can be used in different forms of cells: triangular, rectangular, and hexagonal cells. These different forms of cells divide the plane, generating different tesselations.

3.1.1. The perimeter of the shape

The perimeter P of a shape composed of cells (for instance, pixels) corresponds to the sum of the lengths of the sides of the closed shape. This perimeter corresponds to the classical concept of perimeter. For example, the perimeter of the shape represented in Figure 2a is 14. This perimeter is marked with a bold line.

3.1.2. The contact perimeter

The contact perimeter P_c of a shape composed of cells corresponds to the sum of the lengths of segments which are common to two cells. For instance, the contact perimeter of the shape represented in Figure 2b is 11, this perimeter is composed of 11 segments and is marked with bold lines.

3.2. The Relation Between the Contact Perimeter and the Perimeter of the Shape

THEOREM. *For any shape S_n composed of n cells. The following equation is satisfied:*

$$2P_c + P = Tn, \tag{1}$$

where P_c is the contact perimeter, P is the perimeter of the shape, and T is the number of the sides of the cell. Geometrically, it means that the sum of two times the contact perimeter plus the perimeter is equal to the total sum of the perimeters of all the cells from the shape.

PROOF. For the base case, when the shape has only one cell we have: $n = 1$; $P_c = 0$; and $P = T$. By direct calculation, using equation (1), we have

$$\begin{aligned} 2 \times 0 + T &= T \times 1, \\ T &= T. \end{aligned}$$

Now, let S_{n+1} be a new shape composed of the shape S_n plus a new cell; l be the number of the contact sides of this new cell; and P_c' and P' be the corresponding perimeters of the shape

S_{n+1}. Therefore,

$$P_c' = P_c + l,$$
$$P' = P - l + (T - l) = P + T - 2l,$$

in the shape S_{n+1} the new contact perimeter is increased by l. The perimeter of the shape S_{n+1} is decreased by l and increased by $(T - l)$, which corresponds to the contribution of the new cell to the perimeter.

Representing equation (1) in terms of P_c' and P', we have

$$\begin{aligned} 2P_c' + P' &= 2(P_c + l) + P + T - 2l \\ &= 2P_c + P + T. \end{aligned}$$

From equation (1): $2P_c + P = Tn$. Therefore,

$$2P_c' + P' = Tn + T,$$
$$2P_c' + P' = T(n + 1). \qquad \blacksquare$$

Using equation (1), the contact perimeter is defined as follows:

$$P_c = \frac{(Tn - P)}{2}. \tag{2}$$

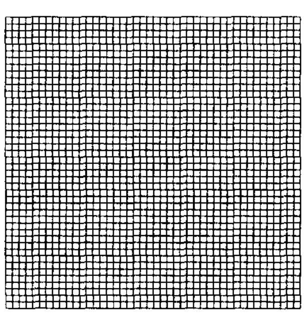
In the case shown in Figure 2b: $n = 9$; $T = 4$; and $P = 14$. Substituting these values in equation (2): $P_c = 11$, which corresponds to the geometrical form.

Figures 2c and 2d show the perimeters for a triangular tesselation. In this case $n = 9$; $T = 3$; and $P = 11$. Therefore, $P_c = 8$. Similarly, in Figures 2e and 2f, the values of the variables are: $n = 9$; $T = 6$; and $P = 26$. Therefore, $P_c = 14$, which corresponds to the sum of all the segment lengths of the shape presented in Figure 2f.

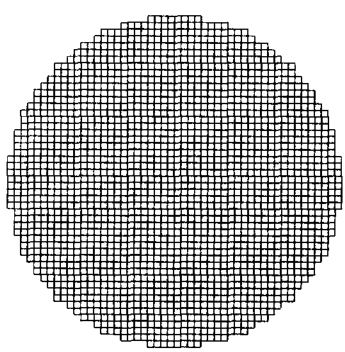
The ratio of perimeters is given by the following equation:

$$\frac{P_c}{P} = \frac{1}{2}\left(\frac{Tn}{P} - 1\right). \tag{3}$$

The contact perimeter may also be obtained for *shapes with holes*. This is presented in Figure 2g, where the perimeter is composed of two contour lines (represented by bold lines) with a total length of 24. Thus, $P = 24$, $n = 12$, and $T = 4$. Therefore, $P_c = 12$.



n=1936
C_D=3784
(a)



n=1936
C_D=3772
(b)

Figure 3. Shapes: (a) a square composed of 1936 pixels; (b) a digital circle composed of 1936 pixels, too.

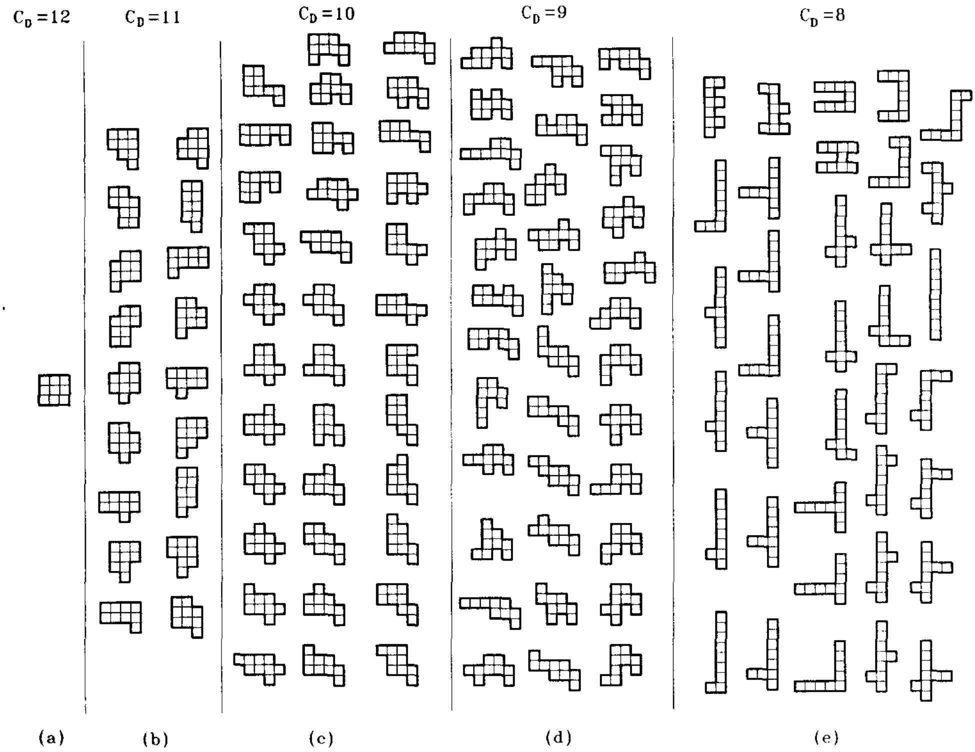


Figure 4. Part of the discrete universe of shapes composed of 9 pixels: (a) the shape having the maximum measure of discrete compactness, its value is equal to 12; (b) shapes which have measures equals to 11; (c) shapes which have measures equals to 10; (d) shapes which have measures equals to 9; (e) shapes having the minimum measure of discrete compactness, its value is equal to 8.

Figures 2h and 2i show *shapes without contact perimeters*. In Figure 2h, $P = 16$, $n = 4$, and $T = 4$. Therefore, $P_c = 0$. Note that in Figure 2h there is no contact perimeter. Similarly, in Figure 2i, where: $P = 20$; $n = 5$; and $T = 4$. Therefore, $P_c = 0$. Notice that in this figure, the pixels are isolated. When the contact perimeter is equal to zero the equation (1) will be

$$P = Tn. \tag{4}$$

4. DISCRETE COMPACTNESS

DEFINITION. *The measure of discrete compactness C_D for a shape composed of n cells corresponds to its contact perimeter, i.e.,*

$$C_D = P_c. \tag{5}$$

In the digital domain, the measure of discrete compactness is maximized to the form of the used cell or a diamond. For example, if the shapes are described using pixels ($T = 4$): the measure of discrete compactness is maximized by a square, Figure 3 shows this. In Figures 3a and 3b, we present a square and a digital circle composed of 1936 pixels each one, and the maximum measure of compactness belongs to the square. This may also be proved using the city block distance ($d_{cb}(x, y) = |x_1 - x_2| + |y_1 - y_2|$). In order to obtain a good measure of compactness, it is important to select an appropriate form of cell. The pixels have a structural problem called the *connectivity paradox*. There are two ways of connecting pixels: *four-connectivity* and *eight-connectivity*. In the content of this paper, we use pixels (i.e., $T = 4$) with four-connectivity.

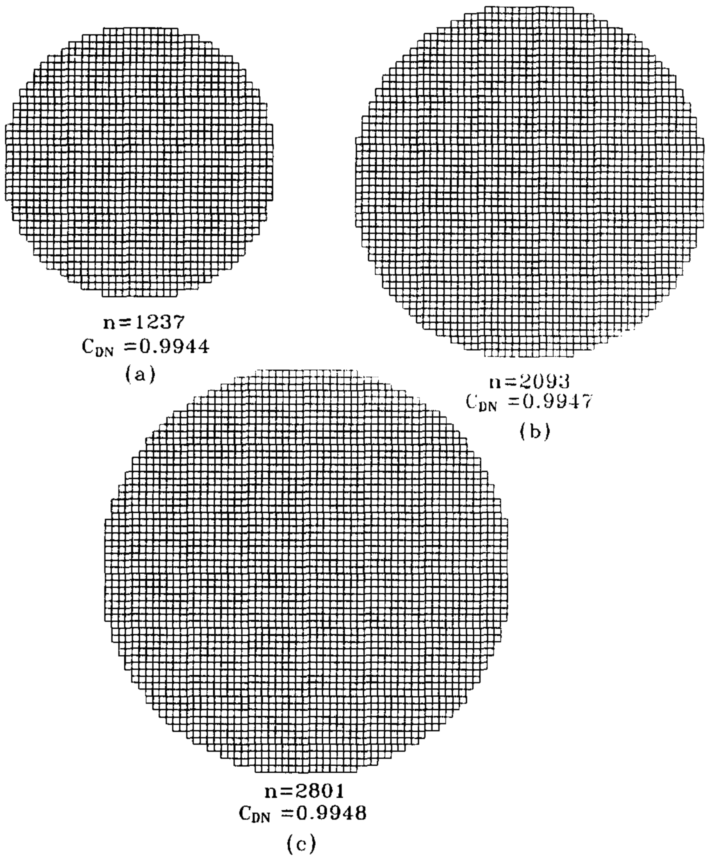


Figure 5. Examples of three different levels of resolution for a digital circle: (a) the digital circle composed of 1237 pixels; (b) the digital circle composed of 2093 pixels; (c) the digital circle composed of 2801 pixels.

4.1. The Minimum and Maximum Measure of Discrete Compactness

The minimum measure of discrete compactness $C_{D\,\min}$ for a shape composed of n pixels is defined by

$$C_{D\,\min} = n - 1. \tag{6}$$

On the other hand, the maximum measure of discrete compactness $C_{D\,\max}$ for a shape composed of n pixels is obtained using equation (1), and is defined by

$$C_{D\,\max} = \frac{Tn - 4\sqrt{n}}{2}. \tag{7}$$

Figure 4 shows part of the discrete universe of shapes composed of nine pixels. Note that the maximum measure corresponds to the shape in Figure 4a, in a contrary direction the minimum measure corresponds to the shapes shown in Figure 4e. Figures 4a–e show the progressive measures of compactness 8, 9, 10, 11, and 12, respectively.

4.2. How to Make the Measure of Discrete Compactness Invariant under Scaling

The measure of discrete compactness should be an intrinsic property of objects. Therefore, it should be invariant under translation, rotation, and scaling. In the digital domain, the measure here proposed of discrete compactness depends on the number of the pixels used to the shape. In order to make the measure of discrete compactness invariant under scaling, we defined the measure of *normalized discrete compactness*, which permits to preserve the discrete compactness

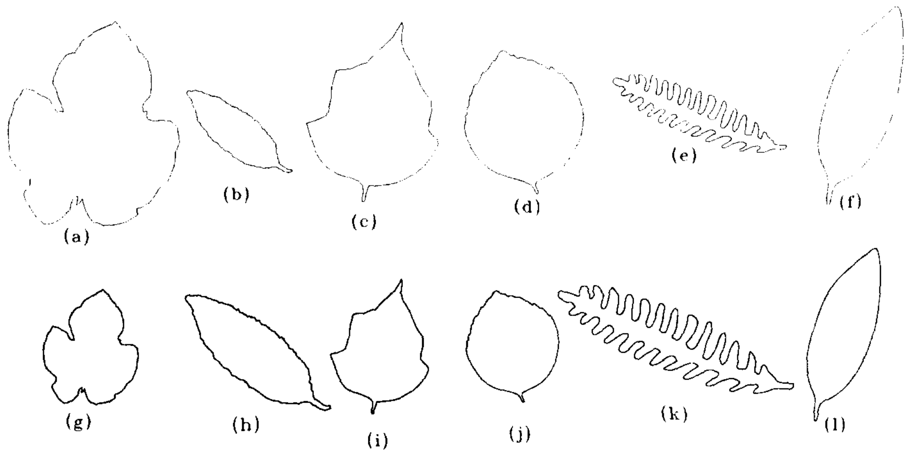


Figure 6. Examples of plant and tree leaves: (a); (b); (c); (d); (e); (f). Shapes invariant under area: (g); (h); (i); (j); (k); (l). These shapes have the same area and correspond to the shapes in the top part.

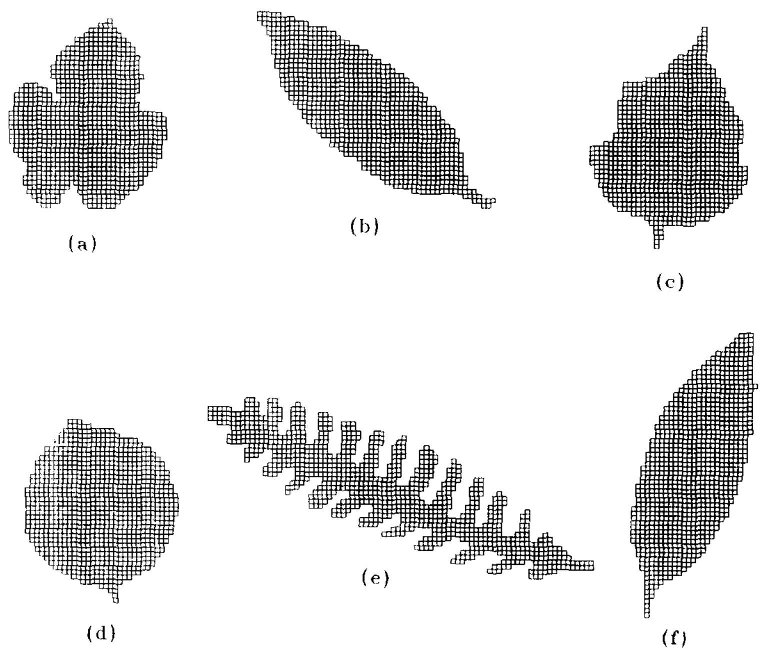


Figure 7. Shapes composed of 900 pixels each one: (a); (b); (c); (d); (e); (f); correspond to shapes shown in the Figures 6g–l, respectively.

for a given shape at different levels of resolution, i.e., the same shape represented using different number of pixels. The measure of normalized discrete compactness C_{DN} is defined by

$$C_{DN} = \frac{C_D - C_{D\min}}{C_{D\max} - C_{D\min}}. \tag{8}$$

The values of the normalized discrete compactness varies continuously from 0 to 1. Thus, the value of the minimum measure of normalized discrete compactness for a shape composed of n number of pixels is zero. On the other hand, the value of the maximum measure is one. Figure 5 shows a digital circle at three different levels of resolution. Figure 5a displays the digital circle composed of 1237 pixels and its measure of normalized discrete compactness is equal to 0.9944. Figure 5b shows the same digital circle composed of 2093 pixels and a measure equal to 0.9947. Finally, Figure 5c shows the same circle now composed of 2801 pixels and a measure equal to

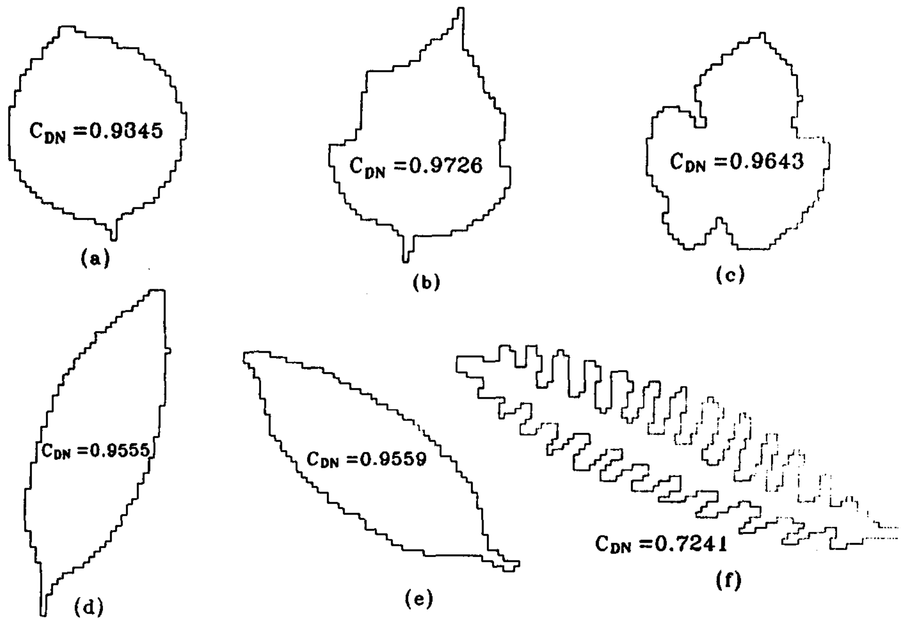


Figure 8. Measures of normalized discrete compactness for the shapes shown in Figure 7. These shapes are presented in descending order of compactness: (a) C_{DN} = 0.9845; (b) C_{DN} = 0.9726; (c) C_{DN} = 0.9643; (d) C_{DN} = 0.9555; (e) C_{DN} = 0.9559; (f) C_{DN} = 0.7241.

0.9948. Notice that the measures of the normalized discrete compactness of the digital circle presented in three different levels of resolution are considered equal. The small differences of the measures are due to n is not a perfect square in the three different cases.

5. RESULTS

When we are classifying shapes to recognize them in the digital domain, the measure proposed here of discrete compactness may be a useful tool. Furthermore, taking into account the *compactness hypothesis* proposed by Haralick and Shapiro [8], which ***"states that the pattern measurements of a given class are nearer to other pattern measurements in the class than they are to pattern measurements from other classes"***. We present some examples using shapes of the real world. These shapes correspond to plant and tree leaves, Figures 6–8 present these examples and the different stages for obtaining their measures of discrete compactness.

Figures 6a–f show the shapes of the leaves, which are represented using straight lines. Notice that these shapes differ in size and orientation. Figures 6g–l represent the Figures 6a–f, respectively, normalized to area. This was obtained by scale changes; the shapes in Figures 6g–l have the same area. Considering $n = 900$: the Figures 7a–f were obtained, which correspond to the Figures 6g–l, respectively.

Finally, Figure 8 presents the measure of normalized discrete compactness for each shape. The shapes are shown in descending order of compactness. For $n = 900$; $C_{D\,\min} = 899$ and $C_{D\,\max} = 1740$. Notice that the shapes in Figures 8d and 8e are similar, the difference between their measures of normalized discrete compactness is very small. On the other hand, Figures 8d and 8e correspond to the Figures 6f and 6b, respectively. The measure of classical compactness of the shape in Figure 6f is: $C = P^2/A = 22.5152$ and the measure of the shape in Figure 6b is: $C = 28.2573$. Observe that the compactness difference between these shapes is important, this difference is due to the noisy contour of the shape in Figure 6b.

6. CONCLUSIONS

In this work, a new perimeter for shapes composed of cells is defined. This definition of perimeter is valid for different forms of cells: triangular, rectangular, and hexagonal cells. The measure proposed here of discrete compactness is based on this perimeter. Also, a relation between the perimeter of the shape and the contact perimeter is presented. Using the normalized discrete compactness, the measure of discrete compactness is invariant under geometric transformations, such as: translation, rotation, and scaling. When pixels are used as cells, we suggest that the number n of cells for pixelization is a perfect square. The measure here proposed of discrete compactness requires more computation than the classical measure, however in some cases, when shapes are represented by pixels may be a useful tool in shape classification. To obtain better results, we recommend the use hexagonal cells, which have *six-connectivity*.

Suggestions for further work: extend the concept of contact perimeter to contact areas for determining the measure of discrete compactness for 3-D shapes composed of voxels.

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